

5 Dilute systems

This chapter considers the behaviour of a single particle suspended in a fluid. In practice, the equations and principles described are used to understand how a number of particles behave, provided that the concentration is sufficiently low enough to ensure that the behaviour of the particle under consideration is not significantly interfered with by the presence of other particles. Applications of the principles covered include particle size analysis by sedimentation methods; where the settling rate is related to the size of the particle, and the industrial process of *clarification* by sedimentation: in which particles are removed from a fluid stream by allowing sufficient time for the particles to settle.

Archimedes' principle
States that when a body is wholly or partially immersed in a fluid it experiences an upthrust equal to the weight of the fluid displaced

5.1 Weight, drag and Particle Reynolds number

All forces must reduce to Newton's basic equation

$$F = ma$$

Forces either cause particle motion in a fluid, or resist it. A force *balance* can be written using all the forces described, or some of these. The easiest force to appreciate is the particle weight, but this is just one example of a *field force*. The particle weight is the product of its mass and the gravitational acceleration. Particles are usually too small to weigh; hence the particle diameter is used to calculate the volume, which is then multiplied by the density to give the mass. Thus, for a spherical particle, the particle weight is (in Newtons)

$$\frac{\pi x^3}{6} \rho_s g \quad (5.1)$$

However, the particle will experience an upward force, in accordance Archimedes' principle, which numerically is

$$\frac{\pi x^3}{6} \rho g \quad (5.2)$$

Hence, combining equations (5.1) and (5.2) provides the *buoyed* particle weight

$$\frac{\pi x^3}{6} (\rho_s - \rho) g \quad (5.3)$$

Before considering other field forces it is illustrative to conduct a simple force balance to see the application of this approach. In a fluid, particle weight will cause an acceleration that will be resisted by fluid drag. When the fluid drag force is equal to the particle weight the motion will be uniform; i.e. no longer accelerating and the particle will attain its *terminal settling velocity*. Fluid drag force comes from a suitable solution to the Navier-Stokes equation. However, this has only been achieved analytically under conditions of no turbulence within the fluid; i.e. streamlines of fluid flowing past the particle, as

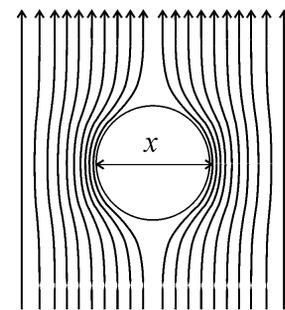


Fig. 5.1 Flow streamlines in fluid around a sphere

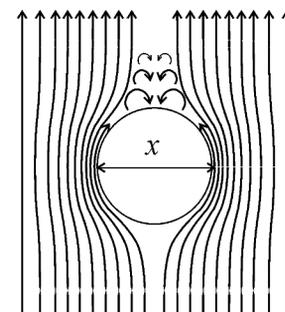


Fig. 5.2 Streamlines and turbulences in a fluid around a sphere - at higher Re'

illustrated in Figure 5.1. Under these conditions Stokes' drag expression is valid

$$F_D = -3\pi U x \mu \tag{5.4}$$

which can be combined with equation (5.3) to provide an expression for terminal settling velocity (U_t), called Stokes' law

$$U_t = \frac{x^2(\rho_s - \rho)g}{18\mu} \tag{5.5}$$

In equation (5.4) the drag force is related to the particle velocity (U), for all values of velocity, whereas in equation (5.5) we are referring to the final (terminal) settling velocity of the particle in a static fluid after the period of acceleration (U_t). Clearly, the settling rate of a particle is a function of its size, solid density and physical properties of the suspending fluid. Equation (5.5) is only valid when the degree of turbulences within the fluid is negligible, see Figure 5.2. This is measured by The Particle Reynolds Number

$$Re' = \frac{xU\rho}{\mu} \tag{5.6}$$

where the threshold for streamline flow past the particle is believed to be about 0.2. The Particle Reynolds Number measures the ratio of inertial to viscous forces within the fluid; hence it is the fluid properties that should be used in it: fluid density and viscosity. At Particle Reynolds Numbers greater than 0.2 the degree of turbulence becomes more significant leading to an additional fluid drag force due to form drag. Hence, the terminal settling velocity will be lower than that predicted by Stokes' law, equation (5.5), which considers only viscous drag around the particle.

In common with fluid flow in pipes, it is possible to correlate a friction factor with Reynolds number for the case of fluid flow past a single spherical particle. This is illustrated in Figure 5.3. The friction factor is the shear stress in a plane at right angles to the direction of motion at the particle surface (R) divided by the fluid density and relative velocity between the particle and fluid squared. The drag force is the product of the shear stress and the particle area, which is the projected area to the fluid flow (A_p). In particle settling it is usual to use a drag coefficient (C_d), rather than friction factor, these are related as follows

$$C_d = \frac{f}{2} = \frac{R}{\rho U^2} \tag{5.7}$$

where the fluid drag (F_D) is

$$F_D = R A_p \tag{5.8}$$

The projected area for a sphere is

$$A_p = \frac{\pi}{4} x^2 \tag{5.9}$$

Galileo (1564-1642)
Galileo is credited with dropping different sized balls from the top of the leaning tower of Pisa to show that they fell at the same rate. This ignores air drag and Galileo knew better than this, he had worked on wind friction. Note that equation (5.5) is not valid for balls in air, but (5.11) is generally valid.

exercise 5.1
Determine the maximum particle size at which Stokes' law should be applied for the data:
solid density: 2500 kg m⁻³
liquid density: 1000 kg m⁻³
viscosity: 0.001 Pa s
i.e. use Re'=0.2 and substitute equation (5.5) in to (5.6).

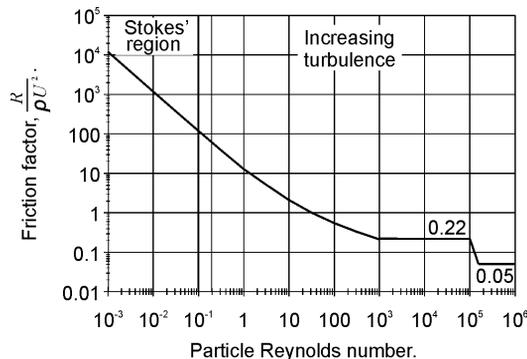


Fig. 5.3 The drag coefficient or friction factor plot for single spherical particles

Combining equations (5.7) to (5.9) provides

$$F_D = C_d \rho U^2 \frac{\pi}{4} x^2 \quad (5.10)$$

Equation (5.10) can be equated with (5.3) to provide a generally valid equation for the terminal settling velocity

$$U_t = \sqrt{\frac{2(\rho_s - \rho)gx}{3\rho C_d}} \quad (5.11)$$

However, equation (5.11) can only be used to predict terminal settling velocity if a value of the drag coefficient is known, see Figure 5.3. In the streamline flow region we know that

$$C_d = \frac{12}{\text{Re}'} \quad (5.12)$$

which can be substituted into equation (5.11) together with (5.6) to provide equation (5.5). Hence, the drag coefficient and Stokes' law approach to particle settling are compatible. However, for Particle Reynolds numbers greater than 0.2 no single and simple analytical function, equivalent to equation (5.12), can be used. Many correlations have been suggested; all of them are valid over a restricted range of Particle Reynolds numbers. An alternative approach comes from considering the drag coefficient further

$$C_d = \frac{2}{3} \frac{(\rho_s - \rho)g}{\rho U_t^2} x \quad (5.13)$$

Equation (5.13) contains both settling velocity and particle size and cannot be used to give the drag coefficient from a diameter because the settling velocity is also required. However, multiplying by the Particle Reynolds number squared results in

$$C_d \text{Re}'^2 = \frac{2}{3} \frac{(\rho_s - \rho)g}{\rho U_t^2} x \frac{x^2 U_t^2 \rho^2}{\mu^2} = \left[\frac{2}{3} \frac{(\rho_s - \rho)g\rho}{\mu^2} \right] x^3 \quad (5.14)$$

The term in the square brackets contains neither particle diameter, nor settling velocity. Likewise, dividing Particle Reynolds number by the drag coefficient gives

$$\frac{\text{Re}'}{C_d} = \frac{x U_t \rho}{\mu} \frac{3}{2} \frac{\rho U_t^2}{(\rho_s - \rho)gx} = \left[\frac{3}{2} \frac{\rho^2}{(\rho_s - \rho)g\mu} \right] U_t^3 \quad (5.15)$$

Equations (5.14) and (5.15) can be written as

$$C_d \text{Re}'^2 = P_H^3 x^3 \quad \text{and} \quad \frac{\text{Re}'}{C_d} = \frac{U_t^3}{Q_H^3}$$

where both P_H and Q_H are not dependent upon particle size or settling velocity. The friction factor correlation, Figure 5.3, can then be redrafted in these terms to give Figure 5.4. So, for the purposes of determining the settling velocity from a given particle diameter, equation (5.14) provides a value for $C_d \text{Re}'^2$, Figure 5.4 is then used to find Re'/C_d and equation (5.15) to provide the settling velocity.

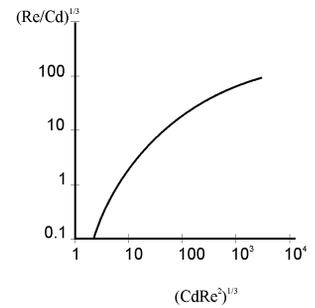
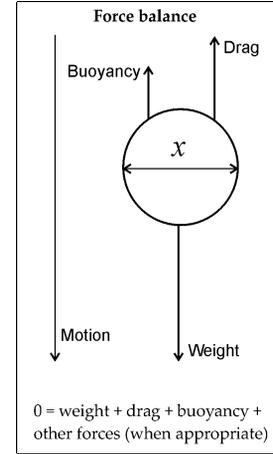


Fig. 5.4 Modified drag and Reynolds number plot

In practice Figure 5.4 is not very easy to use: it is a logarithmic plot and the resolution reduces as a decade is approached. Thus it may be easy to read off an unambiguous value at 11, but difficult to read off a value at 90. To overcome this problem a set of tables was produced by Heywood, correlating $\log_{10}(P_{HX})$ against $\log_{10}(U_t/Q_H)$ and vice-versa, see the Appendix. Thus, in order to determine the settling velocity from a particle diameter, $\log_{10}(P_{HX})$ is first calculated and used to determine $\log_{10}(U_t/Q_H)$ from The Heywood Tables. This value is then anti-logged and multiplied by Q_H to give the velocity. Clearly, this procedure is only worth the extra computational effort when the settling is at Particle Reynolds numbers greater than 0.2. See the box on page 50 for an example of how to use the tables.

The main advantage of the Heywood Tables approach, over empirical correlations between the Particle Reynolds number and a derived function of the drag coefficient, is that it is valid for all Particle Reynolds numbers. It is also possible to implement on a computer and is available *via* the Internet at:

www.filtration-and-separation.com/settling

An alternative popular correlation, using the Particle Reynolds number and The Archimedes (Ar) number, is

$$Re' = \left[\left(14.42 + 1.827 Ar^{0.5} \right)^{0.5} - 3.798 \right]^2 \quad (5.16)$$

valid for $2 < Re' < 20\,000$, where The Archimedes number is closely related to equation (5.14) and is

$$Ar = \left[\frac{(\rho_s - \rho)}{\rho} \right] \frac{g \rho^2 x^3}{\mu^2} \quad (5.17)$$

Thus, to determine a settling velocity the Archimedes number is deduced from equation (5.17), followed by the Particle Reynolds number by equation (5.16) and hence the velocity from (5.6).

5.2 Other forces on particles

Field forces other than the gravitational include

$$\text{centrifugal: } F = \frac{\pi x^3}{6} (\rho_s - \rho) r \omega^2 \quad (5.18)$$

electrical, thermophoretic (due to a temperature gradient), thermal creep (due to greater loss of molecules from the hotter side of a particle), and photophoretic (due to a light intensity gradient). The centrifugal field force is considered further in Chapter 8, electrical and thermophoretic forces in 14 and colloidal forces in 13. The *inertial force* (F_i) is the rate of change of momentum

$$F_i = \frac{\pi}{6} x^3 \rho_s \frac{dU}{dt} \quad (5.19)$$

note that the product of volume and particle density has been used for mass, assuming a spherical particle.

The fluid drag force may be subject to the *mean free path correction*, which is required when the particle size is comparable to the mean free path of the fluid. This is required because the particles can slip between the fluid molecules - effectively reducing the viscous drag. It is more prevalent when the fluid is a gas. The correction to the drag coefficient is

$$C_d = C_{d(\text{STOKES})} [1 + 1.7\lambda / x]^{-1} \quad (5.20)$$

where λ is the mean free path length of the gas. For air $\lambda = 0.1 \mu\text{m}$ (approximately) so the error in assuming the Stokes drag term is 17% for a $1 \mu\text{m}$ particle and 170% for a $0.1 \mu\text{m}$ particle settling in air.

When particles come to rest on each other, or a surface, there is a solids *stress gradient*, or reaction force. This force can be rationalised by considering when a particle is at rest at the base of a vessel, as it experiences no drag or inertia, but still possesses a weight (field) force. This force must be balanced as the particle does not accelerate. The reaction force is due to a pressure, or stress gradient, exerted from the vessel base.

5.3 Particle acceleration in streamline flow

In the derivation of equations (5.5) and (5.11) the drag force was equated to the gravitational field force, to determine the terminal settling velocity of the particle. This simple force balance is only valid if the particle inertial forces can be neglected. Therefore, the time taken to reach the terminal settling velocity, or 99% of it, is a useful check on the validity of the simple force balance used to derive these equations. A force balance of the apparent mass (buoyed mass), drag and inertia for a spherical particle is

$$\frac{\pi x^3}{6}(\rho_s - \rho)g - 3\pi\mu xU - m_p \frac{dU}{dt} = 0 \quad (5.21)$$

where m_p is the actual particle mass, not buoyed mass. Equation (5.21) can be rearranged to give

$$\frac{dU}{dt} = \frac{\pi(\rho_s - \rho)x^3g / 6 - 3\pi\mu xU}{m_p} \quad (5.22)$$

but

$$\frac{\pi}{6}(\rho_s - \rho)g = 3\pi\mu xU_t \quad (5.23)$$

i.e. the equation for the terminal settling velocity.

Therefore,

$$\frac{dU}{dt} = \frac{3\pi\mu xU_t}{m_p} (1 - U/U_t) \quad (5.24)$$

We need to integrate equation (5.24) to find time taken to reach a given velocity, or fraction of terminal settling velocity. The equation can be rearranged to give

$$\int \frac{-1/U_t}{1 - U/U_t} dU = -\frac{3\pi\mu x}{m_p} \int dt$$

Brownian motion

When small particles are suspended in liquids, they are subject to molecular bombardment giving rise to Brownian motion. Hence, finely divided particles may not settle. In practice, particles smaller than $2 \mu\text{m}$ suspended in water will settle slower than predicted by Stokes' law and particles less than $1 \mu\text{m}$ might not settle at all.

i.e. a mathematical relation of the form:

$$\int \frac{f'(x)}{f(x)} = \ln[f(x)]$$

Therefore,

$$\left[\ln(1 - U / U_t) \right]_{U=0}^{U=U_t} = \left[\frac{-3\pi\mu x}{m} t \right]_{t=0}^{t=t}$$

the constant of integration is zero. Hence,

$$t = -\frac{m_p}{3\pi\mu x} \ln(1 - U / U_t) \tag{5.25}$$

where the actual mass of particle is m_p :

$$m_p = \frac{\pi}{6} x^3 \rho_s$$

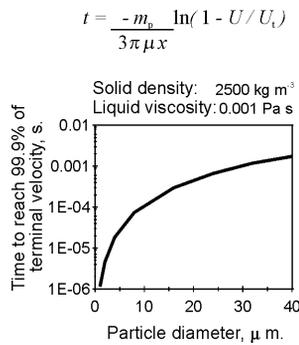


Fig. 5.5 Time taken to reach 99.9% of terminal settling velocity – for the conditions illustrated

On considering equation (5.25) it should be apparent that the particle will never reach its terminal settling velocity: it asymptotes to this value. However, most small particles that are encountered within Particle Technology will reach 99.9% of their terminal settling velocity within a very short acceleration time. See Figure 5.5 for an illustration of this.

5.4 Settling basin design (Camp-Hazen)

Figure 5.6 illustrates the principle behind continuous settling basin design, using a rectangular clarifier. The feed flow enters the vessel on the left and plug flow conditions are assumed, with treated effluent leaving on the right of the vessel. Whilst inside the vessel particles sediment and if they reach the base of the vessel, before being removed in the effluent, then the particles are assumed to stick to the base and be removed from the liquid. Hence, the vessel design requirement is to allow sufficient residence time within the vessel to provide adequate particle removal. The design is based on the *critical trajectory model*: where a particle size is selected and a balance undertaken equating the time taken for the particle to settle the full basin height, and the residence time within the basin assuming plug flow. The resulting simple vector analysis of the trajectory is a straight diagonal line: starting at the top left of the vessel and finishing at the bottom right. All particles of this size will be collected; as those starting their trajectory from further down than the full vessel height (H) will follow a parallel trajectory to the critical and, therefore, reach the vessel base before the full vessel length (L). The time taken to settle will be

$$t_s = \frac{H}{U_t} \tag{5.26}$$

and the residence time, assuming plug flow, is

$$t_r = \frac{HWL}{Q} \tag{5.27}$$

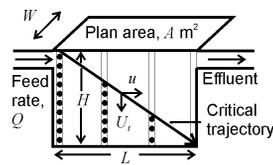


Fig. 5.6 Critical trajectory model for continuous settling basin design

where W is the vessel width (i.e. residence time is vessel volume divided by volumetric flow rate). Equating these two times, cancelling and rearranging provides

$$\frac{Q}{U_t} = LW \tag{5.28}$$

The product of the vessel length and width is the *plan area*. Hence, for complete removal of particles of a given size, the volume flow rate divided by the corresponding terminal settling velocity is equal to the plan area for the complete removal. If the plan area is too small then not all the particles of the selected size will be removed. In settling it is always the plan area that is the important design parameter and not the vessel cross-sectional area. This important fact will be met again and in all cases the provision of a too small plan area will result in incomplete settling, or particle removal.

Further consideration of equation (5.28) and (5.5) provides an indication of the efficiency of removal of particles smaller than the critical size. All particles larger than the critical will sediment out in the available time, and the fraction of smaller ones removed will be directly proportional to the settling velocity - assuming that the feed flow is in fact uniformly distributed over the height of the vessel and not all entering the vessel at the top. Hence, particles half the critical size will only be collected with 25% efficiency because the settling velocity is proportional to the particle diameter squared.

5.5 Laboratory tests

Practical laboratory tests to deduce settling parameters for the design of industrial clarifiers involve the *short tube* and *long tube* tests. Figure 5.7 illustrates the long tube test, where the suspension is allowed to settle within the tube for a set time and the contents above a sample point are drained off. The concentration above the sample point is determined by weighing and drying. For each height it is possible to plot the concentration remaining in suspension against inverse time, as illustrated in Figure 5.8. It may be possible to extrapolate this plot to an inverse time value of zero; which will represent the concentration of unsettleable solids. These are fine particles that remain suspended due to molecular bombardment, or colloidal repulsion forces. Assuming a plot similar to Figure 5.8 provides a suspended solids concentration that is acceptable for an effluent from a continuous clarifier, then the required settling time and height of the sample point (measured downwards from the suspension top), are used in the following equation for vessel plan area

$$A = \frac{Q}{H/t} \frac{1}{E_A} \tag{5.29}$$

where E_A is an area efficiency to take into account turbulences, poor flow distribution, etc. within the vessel.

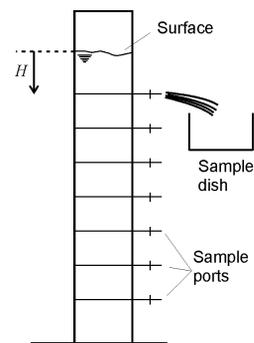


Fig. 5.7 The long tube test

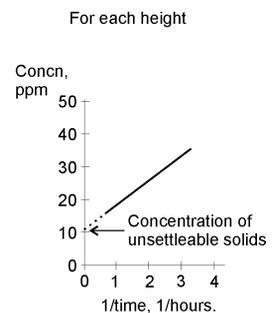


Fig. 5.8 Results from long tube test

5.6 Summary

The settling velocity of small particles may be reliably obtained from Stokes' law. Larger particles, however, do not obey Stokes' law. Alternative correlations between drag coefficient and Particle Reynolds number do exist - but the settling velocity is a constituent of the Particle Reynolds number; hence the answer needs to be known before the appropriate equation to use can be identified! To overcome this problem Heywood published a set of tables that can be used over a wide range of settling velocities and particle sizes.

The single particle settling discussed in this chapter is widely used in engineering calculations. For example, within a spray drier trajectory analysis is often performed using the drag coefficient and the difference in velocity between the particle and the gas is of use in mass transfer calculations. Single particle settling also forms the basis for understanding the behaviour of more concentrated dispersions, which is the subject of the next chapter.

Heywood Tables

(see Appendix)

$$P_H = \left[\frac{4(\rho_s - \rho)\rho g}{3\mu^2} \right]^{1/3}$$

and

$$Q_H = \left[\frac{4(\rho_s - \rho)\mu g}{3\rho^2} \right]^{1/3}$$

Both functions are size and velocity independent. When calculating the settling velocity given a particle diameter (x) the value of $\log(P_H x)$ is first calculated. Then the first two significant figures of $\log(P_H x)$ are given by the first column of the table, the second and third come from the scale given at the top of the table (the first row). The corresponding value of $\log(U_i/Q_H)$ is then read or estimated from the table and converted into a value for U_i using the calculated function Q_H .

5.7 Problems

1.

i). A solid and liquid has a specific gravities of 2.8 and 1.0, respectively and the liquid viscosity is 0.001 Pa s, the value of the function P_H is $2.87 \times 10^4 \text{ m}^{-1}$, the value for Q_H is (SI units):

- a: 2.87×10^{-2} b: 2.87×10^4 c: 2.87 d: 0.490

ii). The SI units of the function Q_H are:

- a: m^{-1} b: m s^{-1} c: m s^{-2} d: s m^{-1}

iii). Use the Heywood Tables to complete the following:

Particle diameter (μm):	1	10	50	100	1000
$\log(P_H x)$:	-----	-----	0.		
$\log(U_i/Q_H)$:	-----	-----			
Settling velocity* (m s^{-1})	-----	-----			
Stokes' settling velocity (m s^{-1}):					

* settling velocity using the Heywood Tables

iv). Why should the Stokes' settling velocities of the larger particles always be greater than those found in practice (and given by the Heywood Tables)?

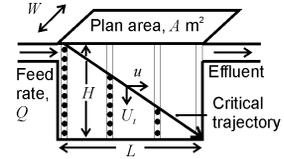
v). The Particle Reynolds is defined as $Re' = \frac{xU_t \rho}{\mu}$

which should be below some threshold for Stokes' law to be applicable. The maximum particle size at which Stokes' law is applicable for the above system is (μm):

- a: 59 b: 5900 c: 127 d: 1271

2.

i). See the continuous settling basin on the right. What will be the trajectory of particles the same size as the critical particle size, but which start their descent from a height less than H ?



ii). The solid and liquid densities are 2900 and 1000 kg m⁻³, the viscosity is 0.001 Pa s and the critical particle diameter is 50 μm, the terminal settling velocity (U_t) is (m s⁻¹):

- a: 2.6×10^{-3} b: 2.6×10^{-5} c: 2.6×10^{-1} d: 0.52

iii). The Particle Reynolds number is:

- a: 0.26 b: 129 c: 0.129 d: 0.00013

iv). An expression for the critical particle residence time vertically (t_v) is (s):

- a: $t_v = L/U$ b: $t_v = H/U$ c: $t_v = H/v$ d: $t_v = L/U$

v). An expression for the critical particle residence time horizontally (t_h) is (s):

- a: $t_h = LWH/Q$ b: $t_h = H/Q$ c: $t_h = L/U$ d: $t_h = Q/LWH$

vi). An expression for LW is (SI units):

- a: $LW = U/Q$ b: $LW = Q/U$ c: $LW = HQ/U$ d: $LW = Q/HU$

vii). What are the units of LW , and what does it represent?

viii). If the volume flow rate into the basin is 10 m³ min⁻¹ the minimum settling area required to remove all particles of the diameter given in Part (ii), and bigger, is (m²):

- a: 6.4 b: 64 c: 640 d: 6400

3.

i). An effluent containing a mineral in suspension with solid and water densities of 2600 and 1000 kg m⁻³, respectively, is pumped into a *batch* vessel 5 m high and left for 30 minutes prior to discharge into a river. The viscosity of water is 0.001 Pa s. The maximum particle diameter that will be in the discharge is (μm):

- a: 3180 b: 56.4 c: 90 d: 66.4

ii). The effluent solid has the following particle size distribution:

Cumulative mass undersize (%):	100	92	80	62	48	31	18	8	4	0
Particle diameter (μm):	90	80	70	60	50	40	30	20	10	0

If the initial concentration of the effluent before settling was 60 mg l⁻¹, the concentration of solids below the size calculated in your answer in Part (ii) (the critical size) is (mg l⁻¹):

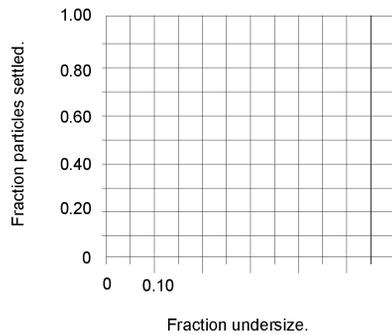
- a: 60 b: 43.2 c: 34.2 d: 25.8

In the Camp-Hazen settling basin model the feed to a basin is assumed to enter well mixed and distributed evenly over the full depth of the vessel. The critical trajectory is given by a particle of a certain (critical) size that enters the basin at the top left and has just settled by the end of the basin length - bottom right. The critical trajectory will be a straight line. Particles smaller than the critical size will also have a straight line trajectory but one that does not intercept with the base by the time the fluid element has reached the end of the basin.

This represents a 'worst case' estimate of the concentration in the effluent discharge after settling, as it assumes that no solids smaller than the critical size settle in the allowed time.

iii). Of the concentration of solids below the critical size a considerable fraction will also have settled out. The amount settled out at each particle diameter is proportional to the ratio of its settling velocity compared to the velocity of the critical particle. For example, a particle with a settling velocity half that of the critical particle will travel 2.5 m in the allowed 30 minutes and, if we assume the suspension was homogeneous before settling, half of the solids at that diameter will settle out. Complete the following table:

Diameter (μm):	70	60	56.4	50	40	30	20	10	0
Fraction settled at size:	1.00								
Fraction undersize:	0.80	0.62	0.57						



iv). Now, to estimate the amount of material settled below the critical size a plot of fraction of particles settling in allowed time against fraction of material undersize is made and the area under the curve is calculated by graphical means. Plot these on the left.

The area under the curve is:
 a: 0.133 b: 0.265 c: 0.53

This represents the fraction of the total distribution below the critical size but which still settles because the particles still reach the base of the vessel in 30 minutes. Add this fraction to the fraction of material in the size distribution above the critical size (which has all settled out). See your answer to Part (ii) to help you find this. NB the next question does not want the fraction settled – it asks for the effluent concentration going to discharge.

v). Hence, the concentration in the effluent discharge is (mg l^{-1}):
 a: 9.1 b: 43.2 c: 34.2 d: 18.3

vi). If the effluent discharge consent limit is 20 mg l^{-1} , will you be able to discharge this suspension?